Phase Effects in the Nonlinear Interaction of "Negative"-Energy Waves

F. ENGELMANN and H. WILHELMSSON *

Laboratori Gas Ionizzati (Associazione EURATOM-CNEN), Frascati (Rome), Italy

(Z. Naturforsch. **24 a**, 206—207 [1969]; received 30 September 1968)

Nonlinear interaction of "negative"-energy waves is studied assuming well-defined phases and perfect matching conditions. The results are compared with those of the random-phase approach, and significant differences in the dynamics due to phase effects are pointed out.

The lowest-order nonlinear interaction of three monochromatic waves, with well-defined phases, having the frequencies ω_0, ω_1, ω_2 where

$$\omega_0 = \omega_1 + \omega_2 \tag{1}$$

can be shown to be described by

$$du_0/d\tau = -s_1 s_2 u_1 u_2 \cos \Phi\,, \tag{2 a}$$

$$du_1/d\tau = s_0 s_2 u_0 u_2 \cos \Phi\,, \tag{2 b}$$

$$du_2/d\tau = s_0 s_1 u_0 u_1 \cos \Phi\,, \tag{2 c}$$

$$\frac{d\Phi}{d\tau} = \left[s_1 s_2 \frac{u_1 u_2}{u_0} - s_0 s_2 \frac{u_0 u_2}{u_1} - s_0 s_1 \frac{u_0 u_1}{u_2} \right] \sin \Phi\,. \tag{3}$$

The $u_j > 0$ are the moduli of the wave amplitudes,

$$\Phi \equiv \Phi_0 - \Phi_1 - \Phi_2 \tag{4}$$

with Φ_j the phase of the j-th amplitude, and

$$s_j \equiv \text{sign}\left\{ \frac{\partial}{\partial \omega_j} [\omega_j^2 \, \varepsilon(\omega_j)] \right\}, \tag{5}$$

$\varepsilon(\omega_j)$ being the dielectric constants.

From Eqs. (2) and (3) one obtains

$$s_1 u_1^2 - s_2 u_2^2 = m_{12}\,, \tag{6 a}$$

$$s_0 u_0^2 + s_2 u_2^2 = m_{02}\,, \tag{6 b}$$

$$s_0 u_0^2 + s_1 u_1^2 = m_{01}\,, \tag{6 c}$$

$$u_0 u_1 u_2 \sin \Phi = \Gamma \tag{7}$$

with the m's and Γ integration constants, and

$$\frac{d(u_0^2)}{d\tau} = -2 \sqrt{s_1 s_2 [u_0^2 (m_{01} - s_0 u_0^2)(m_{02} - s_0 u_0^2) - \Gamma^2]} \, \text{sign}[\cos \Phi]\,. \tag{8}$$

Two different cases may occur:

I) 1) all s_j are equal, or
2) $s_1 s_2 = -1$ and $s_0 =$ arbitrary;

II) $s_1 s_2 = +1$ and $s_0 = -s_1 = -s_2$
(or, equivalently, $s_0 s_1 = s_0 s_2 = -1$).

Case I has been extensively treated [1-3] with the result that the $u_j(\tau)$ are oscillatory and bounded. Physically, it corresponds to an interaction of waves of positive energy (in an appropriate frame of reference).

We are interested here in case II, for which all $du_j/d\tau$ have the same sign. Hence, all amplitudes vary in the same sense, which is only possible if one of the waves has negative energy (in any frame of reference) [4-6]. Qualitatively, the evolution of the u_j^2 and Φ is readily obtained from the preceding relations (cf. Fig. 1). For $u_j^2 \to \infty$ one has

$$u_j^2 \approx (\tau_\infty - \tau)^{-2} \tag{9}$$

with τ_∞ depending on the initial conditions. Hence, the increase of the wave amplitudes — after a transient period, depending on $|\Phi_0|$ and during which the u_j may even decrease — is faster than exponential [7] and leads to a divergence at a finite τ_∞. At the same time, the phase is locked [7] to $\Phi = \pm\pi$ for sign $\Phi_0 = \pm 1$, irrespective of $|\Phi_0|$. We note that, as in case I, one can solve Eqs. (2) and (3) exactly in terms of elliptic functions [1, 3].

* Permanent address: Institute of Physics, University of Uppsala, Sweden.

1 J. A. ARMSTRONG, N. BLOEMBERGEN, J. DUCUING, and P. S. PERSHAN, Phys. Rev. **127**, 1918 [1962].
2 A. SJÖLUND and L. STENFLO, Appl. Phys. Lett. **10**, 201 [1967]; Physica **35**, 499 [1967].
3 R. SUGIHARA, Phys. Fluids **11**, 178 [1968].
4 B. B. KADOMTSEV, A. B. MIKHAILOVSKII, and A. V. TIMOFEEV, Sov. Phys. JETP **20**, 1517 [1965].
5 V. M. DIKASOV, L. I. RUDAKOV, and D. D. RYUTOV, Sov. Phys. JETP **21**, 608 [1965].
6 M. N. ROSENBLUTH, B. COPPI, and R. N. SUDAN, 3rd Conf. on Plasma Phys. and Contr. Nucl. Fusion Res. Novosibirsk 1968, Paper CN 24/E13.
7 Cf. also P. A. STURROCK, Phys. Rev. Lett. **16**, 270 [1966].

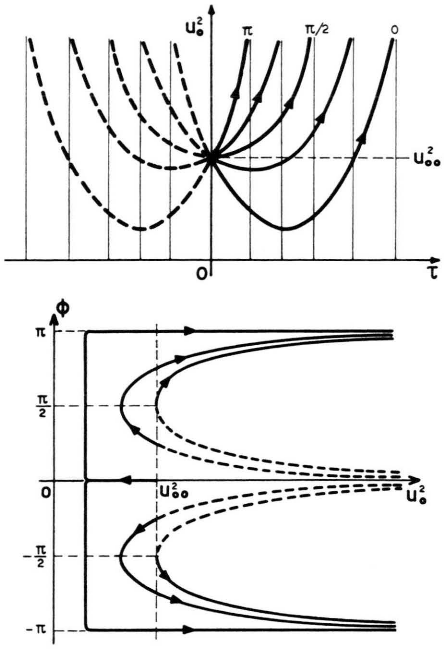

Fig. 1. Qualitative plot of a) u_0^2 as a function of τ with the modulus $|\Phi_0|$ as a parameter, assuming the initial value u_{00}^2 of u_0^2 larger than either u_{10}^2 or u_{20}^2, and b) Φ as a function of u_0^2; the arrows refer to increasing τ.

Let us consider the same interaction problem in random-phase approximation. The corresponding equations of motion are [5, 6, 8, 9]

$$\frac{d(\bar{u}_0^2)}{d\bar{\tau}} = \bar{u}_1^2\,\bar{u}_2^2 - s_0\,s_1\,\bar{u}_0^2\,\bar{u}_2^2 - s_0\,s_2\,\bar{u}_0^2\,\bar{u}_1^2, \qquad (10)$$

and two similar equations for $d(\bar{u}_1^2)/d\bar{\tau}$ and $d(\bar{u}_2^2)/d\bar{\tau}$, from which the validity of Eqs. (6) in terms of the $\bar{u}_j^2$ is easily shown. By using them, exact solutions for the $\bar{u}_j^2(\bar{\tau})$ are readily obtained. For $\bar{u}_j^2 \rightarrow \infty$, one has in case II

$$\bar{u}_j^2 \approx \tfrac{1}{3}\,(\bar{\tau}_\infty - \bar{\tau})^{-1} \qquad (11)$$

with $\bar{\tau}_\infty$ determined by the initial conditions. Hence, the final increase of the wave amplitudes is less "explosive" [7] here than in the case of determined phase. On the other hand, no decrease of the amplitudes, even transitory, is possible in the random-phase case.

In conclusion, phase effects may have considerable importance for the dynamics of the considered wave interaction.

H. WILHELMSSON wishes to thank Prof. B. BRUNELLI for the kind hospitality extended to him at the Laboratori Gas Ionizzati at Frascati.

[8] V. N. TSYTOVICH, Sov. Phys. Uspekhi **9**, 805 [1967].

[9] R. E. AAMODT and M. L. SLOAN, Phys. Rev. Lett. **19**, 1227 [1967].